Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

CHARACTERIZATION OF MAGIC GRAPHS

SAMUEL JEZNÝ and MARIÁN TRENKLER, Košice

(Received January 22, 1982)

I. INTRODUCTION

We shall consider a non-orientable finite graph $G = [V(G), E(G)]$ without loops, multiple edges or isolated vertices. If there exists a mapping f from the set of edges $E(G)$ into positive real numbers such that

(i) $f(e_i) \neq f(e_j)$ for all $e_i \neq e_j$; $e_i, e_j \in E(G)$,

(ii) $\sum_{e \in E(G)} \eta(v, e) f(e) = r$ for all $v \in V(G)$,

where $\eta(v, e) = \begin{cases} 1 & \text{when vertex } v \text{ and edge } e \text{ are incident} \\ 0 & \text{in the opposite case,} \end{cases}$

then the graph G is called *magic*. The mapping f is called a *labelling* of G and the value r is the index of the label f. We say that a graph G is *semimagic* if there exists a mapping f into positive real number which satisfies only the condition (ii). If the semimagic graph G has a label with the index r we shall say that G has index r.

To study magic graphs was suggested by J. Sedláček [3]. Some sufficient conditions for the existence of magic graphs are established in [2], [4] and [5]. A characterization of regular magic graphs in terms of circuits is given by M. Doob [1]. J. Mühlbacher [2] used matrix theory to prove two necessary conditions for the existence of magic graph. These conditions are weaker than that of theorem 2 of this paper.

First we shall formulate several necessary definitions.

A subgraph $F = [V(F), E(F)]$ of the graph $G = [V(G), E(G)]$ is called a *factor* of G if the sets $V(G)$ and $V(F)$ are the same. A factor F is a $(1-2)$-*factor* of G if each of its components is a regular graph of degree one or two. By the symbol F^1, resp. F^2 we denote the subgraph of F which consists of all isolated edges, or of all circuits of F and the necessary vertices, respectively. We say that a $(1-2)$-*factor separates the edge* e_1 and e_2, if at least one of them belongs to F and neither F^1 nor F^2 contains both of them.

The aim of this paper is to characterize all magic graph using the notion of separating edges by a $(1-2)$-factor.

In this part we state some results about semimagic graphs which we shall use to prove the main result.

Lemma 1. *If G is a semimagic graph with the index r, then*

a) *each isolated edge of G has the label r,*

b) *a connected part of G having more than one edge contains no vertex of degree one.*

The proofs of these statements follow from the definition of a semimagic graph.

Lemma 2. *Let a semimagic graph G contain an even circuit C, then there exists a semimagic factor H of G which does not contain all edges of C.*

Proof. Let f be a semimagic labelling of G and let $m = \min\{f(e): e \in E(C)\}$. We denote the edges of C by $e_1, e_2, \ldots, e_{2n}$ and suppose that $f(e_1) = m$. We define a new labelling h of G:

$h(e_{2i-1}) = f(e_{2i-1}) - m,$

$h(e_{2i}) = f(e_{2i}) + m$ for $i = 1, 2, \ldots, n,$

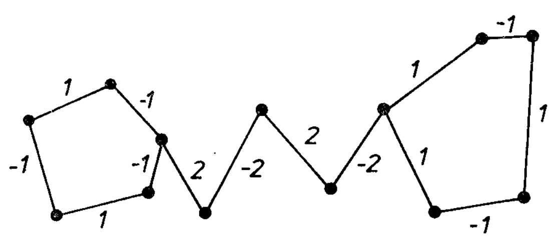
$h(e_j) = f(e_j)$ for all $e_j \notin E(C)$.

Obviously $h(e_1) = 0$. By omitting all edges with $h(e) = 0$ from G we obtain a factor which does not contain all edges of the circuit C and has the same index as the graph G.

The graph D is called a *dumbbell* if it consists of two odd circuits C_1 and C_2 without common vertices joined by a path P or if it consits only of two odd circuits C_1 and C_2 with only one common vertex.

Lemma 3. *Let a semimagic graph G contain as a subgraph a dumbbell D, then there exists a semimagic factor H of G which does not contain all edges of the subgraph D.*

Proof. Let f be a semimagic labelling of G with a dumbbell D which consists of two circuits C_1, C_2 and a path P or only of two circuits C_1, C_2. We denote $m = \min\{m_1, m_2\}$ where $m_1 = \min\{f(e): e \in E(C_1) \cup E(C_2)\}$ and $m_2 = 1/2 \min\{f(e):$



$e \in E(P)\}$. Let e' be an edge of D such that $f(e') = m$. We define an auxiliary labelling p. The edges of C_i have alternating values 1 and -1 and the edges of P the values 2 and -2 such that the sum at each vertex is zero, and the value of the edge e' is negative.

All the other edges of G have value 0. We consider the labelling

$$h(e) = f(e) + m\,p(e) \quad \text{for all} \quad e \in E(G)\,.$$

All edges having $h(e) > 0$ form a semimagic factor H of G which has the same index as G.

From the lemmas 2 and 3 it follows:

Lemma 4. *If G is a semimagic graph, then there exists a semimagic $(1-2)$-factor F of G with the same index.*

Lemma 5. *If G is a semimagic graph, then every edge e' of G is contained in a $(1-2)$-factor.*

Proof. Let e' be an arbitrary edge of G and F some $(1-2)$-factor of G. There are two possible cases: either $e' \in E(F)$ or $e' \notin E(F)$. We must consider only the second case.

Let q be an auxiliary labelling such that

$$\begin{aligned} q(e) &= 2 \quad \text{for all} \quad e \in E(F^1)\,,\\ q(e) &= 1 \quad \text{for all} \quad e \in E(F^2)\,,\\ q(e) &= 0 \quad \text{for all} \quad e \notin E(F)\,, \end{aligned}$$

and

$$m = \min\{f(e)/q(e) \colon e \in E(F)\}\,.$$

We consider a new labelling

$$h(e) = f(e) - m\,q(e) \quad \text{for all} \quad e \in E(G)\,.$$

Omitting from the graph G all edges for which $h(e) = 0$ we obtain a semimagic factor H which contains the edge e'. Let F' be a $(1-2)$-factor of H. (Note that F' is also a $(1-2)$-factor of G.) If $e' \notin E(F')$ we repeat the construction described after. By a finite number of repetitions we obtain a $(1-2)$-factor of G which contains the edge e'.

Lemma 6. *If every edge of G belongs to a $(1-2)$-factor, then G is semimagic.*

Proof. A semimagic labelling of G is obtained by a finite number of repetitions of the following construction.

Let f be a labelling with nonnegative numbers such that the sum of the labels of edges incident with each vertex is the same. (Note that every graph has such a labelling.) Let e be an edge with $f(e) = 0$ and F one $(1-2)$-factor such that $e \in E(F)$. We define a new labelling

$$\begin{aligned} h(e) &= f(e) + 2m \quad \text{for all} \quad e \in E(F^1)\,,\\ h(e) &= f(e) + m \quad \text{for all} \quad e \in E(F^2)\,,\\ h(e) &= f(e) \qquad\qquad \text{for all} \quad e \notin E(F)\,, \end{aligned}$$

where $m = \max\{f(e) \colon e \in E(G)\} + 1$.

From the previous lemmas it follows:

Theorem 1. *The graph G is semimagic if and only if every edge is contained in a* $(1-2)$*-factor.*

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Lemma 7. *If every couple of edges* e_1, e_2 *of a semimagic graph G is separated by a* $(1-2)$*-factor, then G is magic.*

Proof. Let f be a semimagic labelling of G. If $f(e_1) \neq f(e_2)$ for all couples of edges e_1, e_2, then G is magic. In the opposite case we choose a $(1-2)$-factor F which separates e_1 and e_2 and define a new labelling h as in the proof of lemma 6. After a finite number of repetitions of the previous step we obtain a magic graph.

The previous lemmas yield the proof of our main result.

Theorem 2. *A graph G is magic if and only if* (i) *every edge of G belongs to a* $(1-2)$*-factor, and* (ii) *every couple of edges* e_1, e_2 *is separated by a* $(1-2)$*-factor.*

Consequence. *If G is magic graph then there exists a magic labelling of G with positive integers.*

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Authors' address: 041 54 Košice, Jesenná 5, ČSSR (Univerzita P. J. Šafárika).